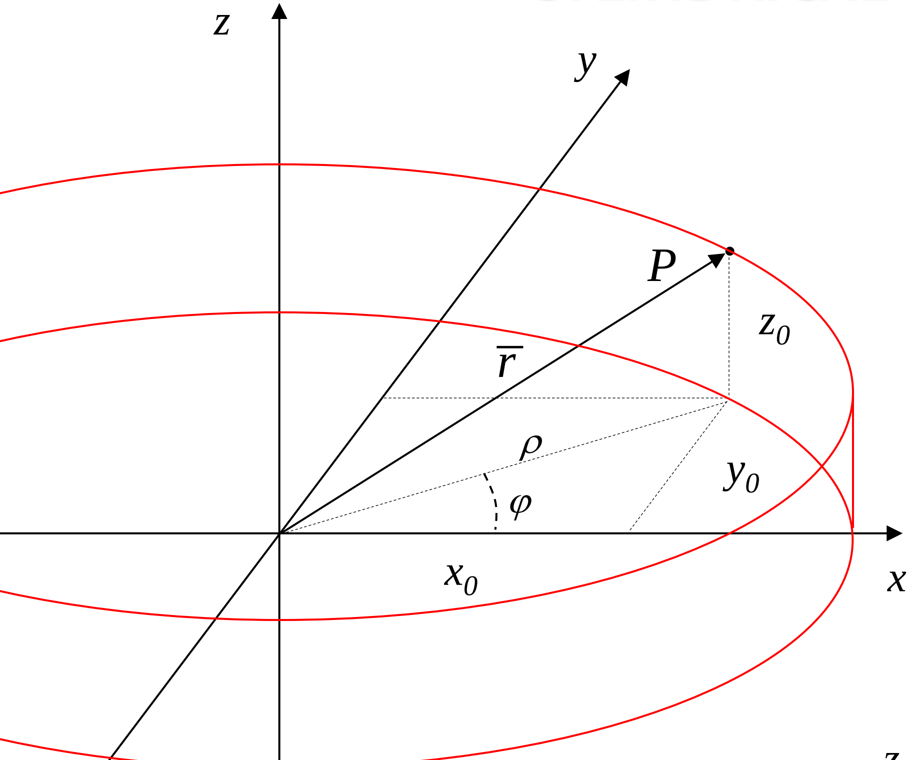
VEKTORANALYS

Kursvecka 3

övningar

CYLINDRICAL COORDINATES

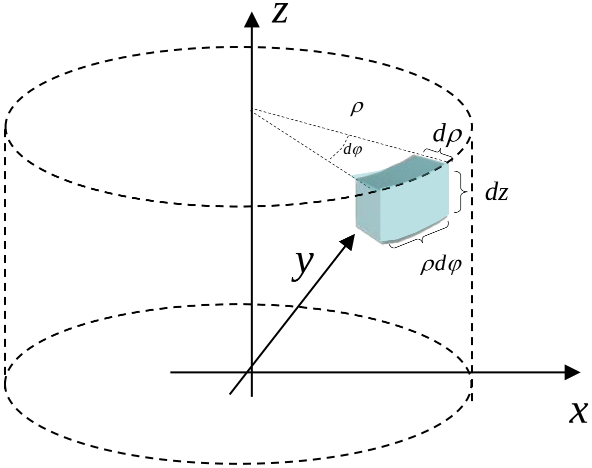


$P: \ (x_0, \ y_0, \ z_0)$ *cartesian coord.*

$P: \ (\rho, \ \varphi, \ z_0)$ *cylindrical coord.*

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases}$$

$$0 < \varphi < 2\pi$$



SURFACE ELEMENT

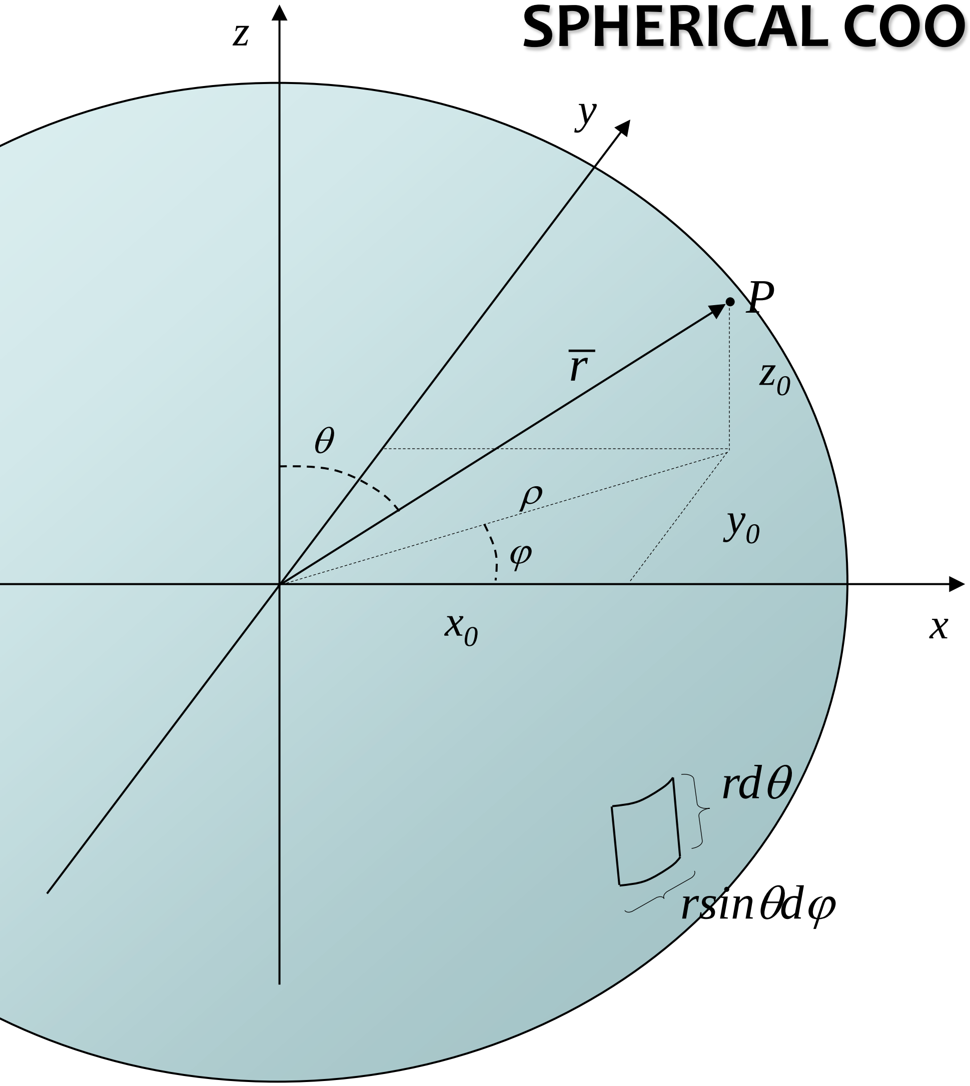
$$d\bar{S} = \hat{e}_\rho \rho d\varphi dz$$ *(on the lateral surface)*

$$d\bar{S} = \hat{e}_z \rho d\varphi d\rho$$ *(on the top and bottom surfaces)*

VOLUM ELEMENT

$$dV = \rho d\rho d\varphi dz$$

SPHERICAL COORDINATES



P: (x_0, y_0, z_0) *cartesian coord.*

P: (r, θ, φ) *spherical coord.*

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases}$$

$$0 < \varphi < 2\pi$$
$$0 < \theta < \pi$$

SURFACE ELEMENT

$$\boxed{d\overline{S} = \hat{e}_r r^2 \sin\theta d\theta d\varphi}$$

VOLUM ELEMENT

$$\boxed{dV = r^2 \sin\theta dr d\theta d\varphi}$$

PROBLEM 1

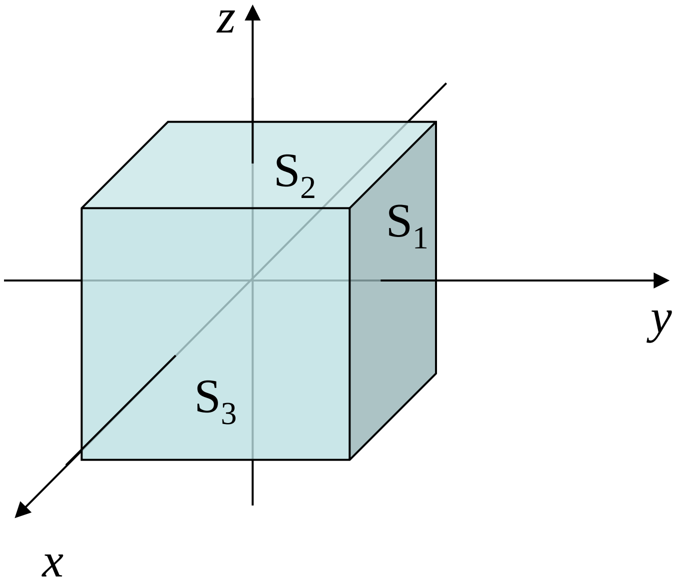
Calculate: $\iint_S \overline{A}\cdot d\overline{S}$ where the vector field is: $\overline{A}=(x,y,z)$

and S is a cube (length 2 each side) centred in the origin.

(a) In a direct way (using the parameterization of the surface)

(b) Using the Gauss theorem

SOLUTION



(a) $$\iint_S \overline{A}\cdot d\overline{S}=\sum_i \iint_{S_i} \overline{A}\cdot d\overline{S}$$

Let's start with S_1

1- parameterization of S_1:

$$\left.\begin{matrix} y=1 \\ |x|<1 \\ |z|<1 \end{matrix}\right\}\Rightarrow \underset{\substack{u:-1\rightarrow+1\\ v:-1\rightarrow+1}}{\overline{r}(u,v)}=(u,1,v)$$

2- Integral calculation:

$$\int_{S_1} \overline{A}\cdot d\overline{S} = \int_u \int_v \overline{A}(\overline{r}(u,v))\cdot\left(\frac{\partial \overline{r}}{\partial u}\times\frac{\partial \overline{r}}{\partial v}\right)dudv$$

$$\left.\begin{array}{l} \dfrac{\partial \overline{r}}{\partial u} = (1,0,0) \\ \dfrac{\partial \overline{r}}{\partial v} = (0,0,1) \end{array}\right\} \Rightarrow \left(\frac{\partial \overline{r}}{\partial u}\times\frac{\partial \overline{r}}{\partial v}\right) = (0,1,0)$$

$$\int_{S_1} \overline{A}\cdot d\overline{S} = \int_{-1}^{1}\int_{-1}^{1}(u,1,v)\cdot(0,1,0)dudv = \int_{-1}^{1}\int_{-1}^{1} dudv = 4$$

Due to the symmetry of the problem we have: $\int_{S_i} \overline{A}\cdot d\overline{S} = 4$

$$\Rightarrow \iint_S \overline{A}\cdot d\overline{S} = \sum_i \iint_{S_i} \overline{A}\cdot d\overline{S} = 6\cdot 4 = 24$$

(b) S is a closed surface $\Rightarrow$ we can apply the Gauss theorem

$$\left.\begin{array}{l} \displaystyle\iint_S \overline{A}\cdot d\overline{S} = \iiint_V div\overline{A}dV \\ div\overline{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z} = 1+1+1 = 3 \end{array}\right\} \Rightarrow \iint_S \overline{A}\cdot d\overline{S} = \iiint_V 3dV = 3V = 3\cdot 2^3 = 24$$

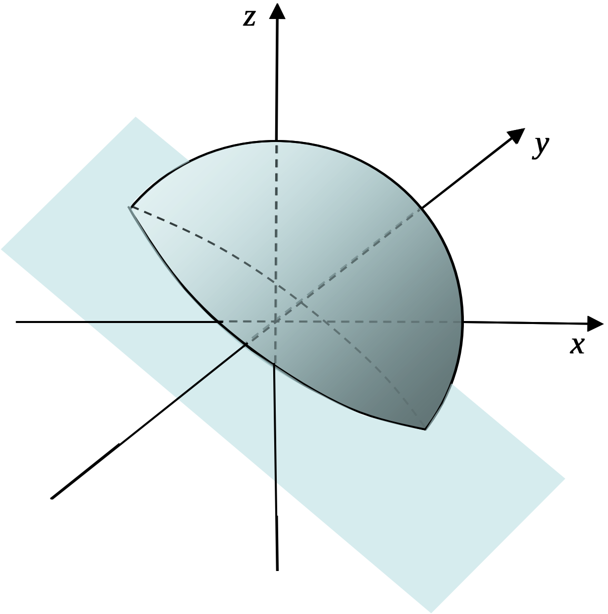
PROBLEM 2

Calculate $\iint_S \overline{A}\cdot d\overline{S}$ using the Gauss theorem

where the vector field is: $\overline{A}=(x^3, y^3, z^3)$

and the surface S is a half sphere defined by: $\begin{cases} x^2+y^2+z^2=R^2 \\ x+y\geq 0 \end{cases}$

SOLUTION



But S is NOT a closed surface!
So we can consider the surface
$S_{tot}=S+S_{plane}$

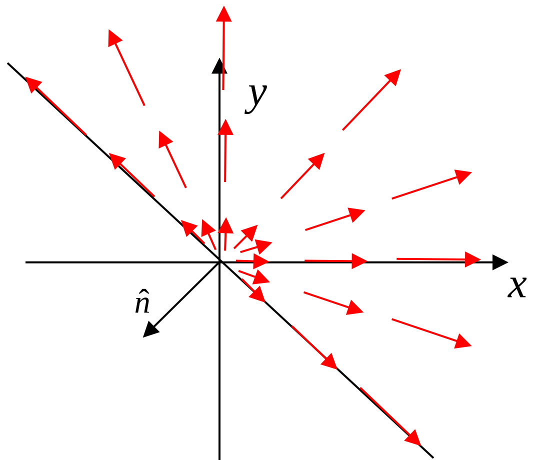
$$\oiint_{S_{tot}} \overline{A}\cdot d\overline{S} = \iiint_V div\overline{A}dV$$

$$\iint_S \overline{A}\cdot d\overline{S} = \oiint_{S_{tot}} \overline{A}\cdot d\overline{S} - \iint_{S_{plane}} \overline{A}\cdot d\overline{S}$$

$$\iint_S \overline{A}\cdot d\overline{S} = \iiint_V div\overline{A}dV - \iint_{S_{plane}} \overline{A}\cdot d\overline{S}$$

So we have transformed a surface integral
into a volume integral minus another surface integral
What is the advantage?
They can be calculated much easier!!

Let’s consider the second integral.



$$S_{plane} \qquad \begin{cases} x^2+y^2+z^2 \le R^2 \\ x+y=0 \end{cases}$$

On S_{plane} $\quad x=-y$

$$\overline{A}=(x^3,y^3,z^3) \qquad \Rightarrow \qquad \overline{A}=(x^3,-x^3,z^3)$$

On S_{plane} the vector is perpendicular to $\hat{n} \quad \Rightarrow \iint_{S_{plane}} \overline{A}\cdot d\overline{S}=0$

Let’s consider the first integral.

Spherical coordinates

$$\iiint_V div\overline{A}dV \quad with \quad div\overline{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=3x^2+3y^2+3z^2 \overset{\downarrow}{=} 3r^2$$

since $div\bar{A} = 3r^2$ $\overset{\textit{due to symmetry}}{\Rightarrow}$ $\iiint\limits_{V} div\bar{A}dV = \frac{1}{2}\iiint\limits_{V_{sphere}} div\bar{A}dV$

$$\iiint\limits_{V_{sphere}} div\bar{A}dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 3r^2\ r^2\sin\theta dr d\theta d\varphi = 3\int_0^{2\pi} d\varphi \int_0^{\pi}\sin\theta d\theta \int_0^{R} r^4\ dr = \frac{12\pi R^5}{5}$$

$$\Rightarrow \quad \iiint\limits_{V} div\bar{A}dV = \frac{1}{2}\iiint\limits_{V_{sphere}} div\bar{A}dV = \frac{6\pi R^5}{5}$$

$$\iint\limits_{S} \bar{A}\cdot d\bar{S} = \iiint\limits_{V} div\bar{A}dV - \iint\limits_{S_{plane}} \bar{A}\cdot d\bar{S} = \frac{6\pi R^5}{5}$$

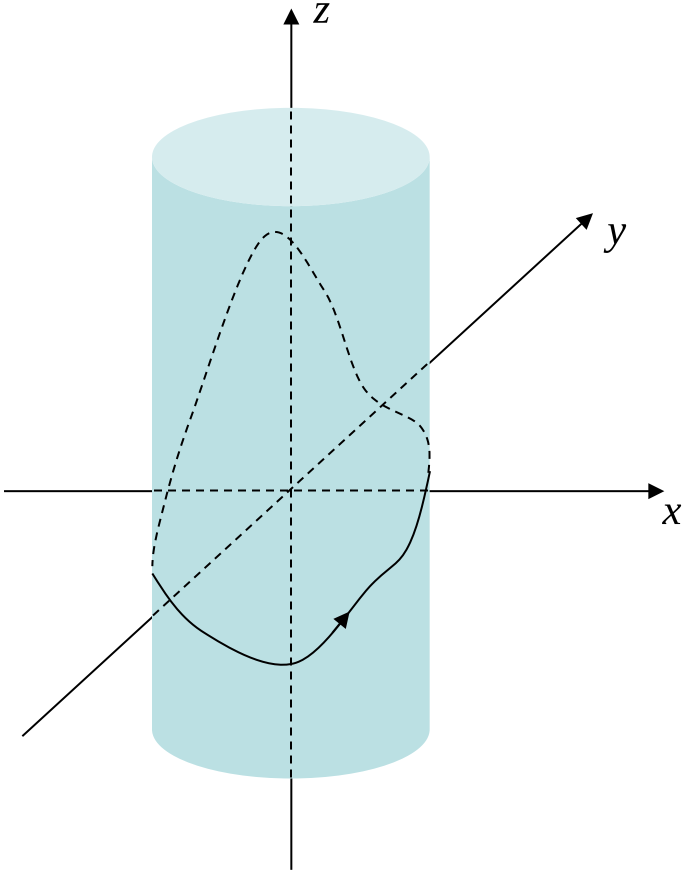
PROBLEM 3

Calculate the line integral of the vector field: $\overline{A} = \left(y + 2x,\ x^2 + z,\ y \right)$

along the closed curve: $\begin{cases} \overline{r}(u) = (\cos u, \sin u,\ f(u)) & \quad with \quad f(0) = f(2\pi) \\ u: \quad 0 \rightarrow 2\pi \end{cases}$

(a) directly
(b) using the Stokes' theorem

SOLUTION



The curve is on the cylinder defined by $(\cos u,\ \sin u,\ z)$

On the cylinder the curve Is defined by $z=f(u)$

SOLUTION (A)

We will calculate $\int_L \overline{A}\cdot d\overline{r} = \int_a^b \overline{A}(\overline{r}(u))\cdot \frac{d\overline{r}}{du}\,du$

$$\frac{d\overline{r}}{du} = \left(-\sin u, \cos u, \frac{df}{du}\right)$$

$$\overline{A}(\overline{r}(u)) = \left(\sin u + 2\cos u, \cos^2 u + f(u), \sin u\right)$$

$$\Rightarrow \int_L \overline{A}\cdot d\overline{r} = \int_0^{2\pi}\left(\sin u + 2\cos u, \cos^2 u + f(u), \sin u\right)\cdot\left(-\sin u, \cos u, \frac{df}{du}\right)du =$$

$$-\int_0^{2\pi}\sin^2 u\,du - 2\int_0^{2\pi}\sin u\cos u\,du + \int_0^{2\pi}\cos^3 u\,du + \int_0^{2\pi}\left(f(u)\cos u + \frac{df}{du}\sin u\right)du =$$

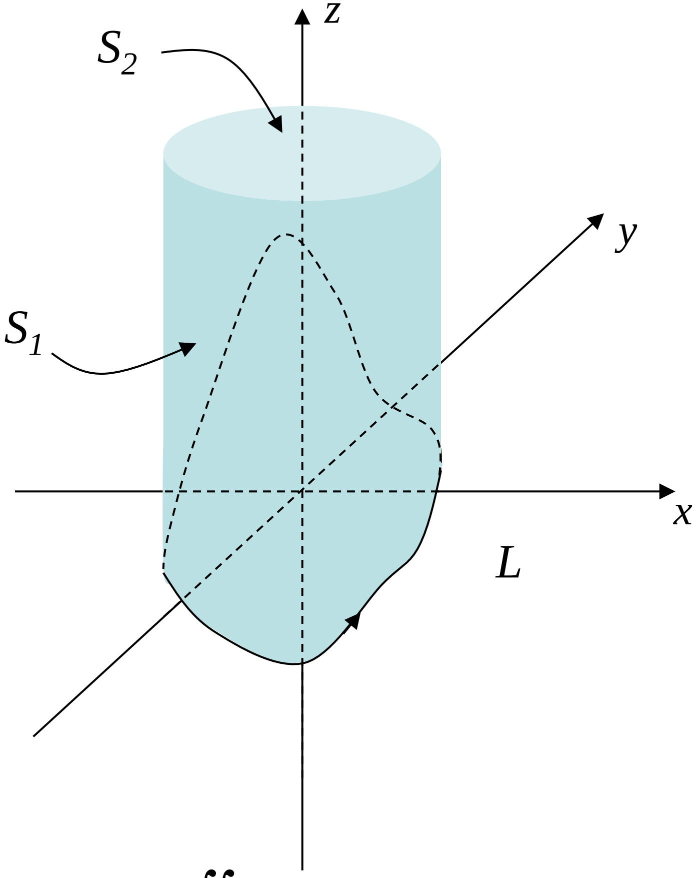
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$-\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{2\pi} - \left[\sin^2 u\right]_0^{2\pi} + \left[\sin u - \frac{1}{3}\sin^3 u\right]_0^{2\pi} + \left[f(u)\sin u\right]_0^{2\pi} = -\pi$$

SOLUTION (B)

$$\int_L \overline{A}\cdot d\overline{r} = \iint_S rot\,\overline{A}\cdot d\overline{S}$$

$S=S_1+S_2$



$$rot\,\overline{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+2x & x^2+z & y \end{vmatrix} = \left(1-1, 0-0, 2x-1\right) = (0,0,2x-1)$$

$rot\,\overline{A}$ is in the z-direction $\Rightarrow \iint_{S_1} rot\,\overline{A}\cdot d\overline{S} = 0$

$$\int_L \overline{A}\cdot d\overline{r} = \iint_S rot\,\overline{A}\cdot d\overline{S} = \iint_{S_2} rot\,\overline{A}\cdot d\overline{S} = \iint_{S_2} (0,0,2x-1)\cdot \hat{e}_z\, dxdy = \iint_{S_2} (2x-1)\,dxdy$$

cylindrical coord.

↓

$$= \int_0^{2\pi}\int_0^1 (2\rho\cos\varphi - 1)\rho\, d\rho\, d\varphi = 2\int_0^{2\pi}\cos\varphi\, d\varphi \int_0^1 \rho^2 d\rho - \int_0^{2\pi} d\varphi \int_0^1 \rho\, d\rho = -2\pi\left[\frac{\rho^2}{2}\right]_0^1 = -\pi$$